3D gravity with torsion as a Chern-Simons gauge theory

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Abstract

We show that topological 3D gravity with torsion can be formulated as a Chern-Simons gauge theory, provided a specific parameter, known as the effective cosmological constant, is negative. In that case, the boundary dynamics of the theory corresponding to anti-de Sitter boundary conditions is described by a conformal field theory with two different central charges.

I. INTRODUCTION

Investigations of the classical structure of three-dimensional (3D) Einstein gravity have had an important influence on our understanding of the related quantum dynamics [1]. In this regard, one should pay a specific attention to the asymptotic structure of 3D gravity [2], as well as to its formulation as a Chern-Simons gauge theory [3,4]. Following a widely spread belief that the dynamics of geometry is to be described by general relativity, investigations of 3D gravity have been carried out mostly in the realm of *Riemannian* geometry [5–8]. Here, we focus our attention on *Riemann-Cartan* geometry, in which both the *curvature* and the *torsion* are present as independent geometric characteristics of spacetime [9,10].

The asymptotic structure is most clearly seen in topological theories, in which the nontrivial dynamics can be present only at the boundary of spacetime. A general action for topological Riemann-Cartan gravity in 3D has been proposed by Mielke and Baekler (MB) [11,12]. The model generalizes general relativity with a cosmological constant (GR_Λ) to a topological gravity in Riemann-Cartan spacetime [13]. A specific version of the MB model, characterized by the teleparallel geometry of spacetime without matter, has been recently used to study the influence of torsion on the asymptotic structure of gravity [14,15]. The results show that both GR_Λ and the teleparallel 3D gravity have identical asymptotic structures (Chern-Simons formulation, conformal symmetry and Liouville dynamics at the boundary), at least at the classical level. Thus, it seems that the asymptotic structure does not depend on the geometric environment, but only on the boundary conditions. In the present paper we extend these investigations to the general MB model in Riemann-Cartan spacetime.

The layout of the paper is as follows. In Sect. II we review the basic features of the MB model for topological 3D gravity in Riemann-Cartan spacetime, and analyze the properties that may be of particular importance for the expected Chern-Simons structure. In Sect. 2

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we show that 3D Riemann-Cartan gravity, in the sector where the so-called effective cosmological constant is negative, can be formulated as a Chern-Simons gauge theory. Section IV concludes our exposition.

Our conventions are the same as in Refs. [14,15]: the Latin indices $(i, j, k, ...)$ refer to the local Lorentz frame, the Greek indices $(\mu, \nu, \rho, ...)$ refer to the coordinate frame, and both run over $0, 1, 2$; $\eta_{ij} = (+, -, -)$ and $g_{\mu\nu} = b^i{}_\mu b^j{}_\nu \eta_{ij}$ are metric components in the local Lorentz and coordinate frame, respectively; totally antisymmetric tensor ε^{ijk} and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized by $\varepsilon^{012} = +1$.

II. TOPOLOGICAL 3D GRAVITY WITH TORSION

In this section we review basic features of the model for topological Riemann-Cartan gravity in 3D proposed by Mielke and Baekler [11,12], and discuss certain dynamical issues which turn out to be essential for identifying its Chern-Simons structure.

A. Topological gravity in Riemann-Cartan spacetime

Riemann-Cartan geometry of spacetime can be formulated as Poincaré gauge theory [9,10]: basic gravitational variables are the triad field $b^i = b^i{}_\mu dx^\mu$ and the Lorentz connection $\omega^{ij} = \omega^{ij}{}_\mu dx^\mu$ (1-forms), and the related field strengths are the torsion T^i and the curvature R^{ij} (2-forms). In 3D we can simplify the notation by introducing the duals of ω^{ij} and R^{ij}:

$$\omega_i = -\frac{1}{2}\,\varepsilon_{ijk}\omega^{jk}\,, \qquad R_i = -\frac{1}{2}\,\varepsilon_{ijk}R^{jk}\,.$$

After that, the gauge transformations of the fields take the form

$$\begin{aligned}
\delta_0 b^i{}_\mu &= -\varepsilon^i{}_{jk} b^j{}_\mu \theta^k - \xi^\rho{}_{,\mu} b^i{}_\rho - \xi^\rho b^i{}_{\mu,\rho} \\
\delta_0 \omega^i{}_\mu &= -\theta^i{}_{,\mu} - \varepsilon^i{}_{jk}\omega^j{}_\mu \theta^k - \xi^\rho{}_{,\mu}\omega^i{}_\rho - \xi^\rho \omega^i{}_{\mu,\rho}\,,
\end{aligned} \tag{2.1a}$$

and the field strengths are given as (wedge product signs are omitted for simplicity)

$$\begin{aligned}
R^i &= d\omega^i + \frac{1}{2}\,\varepsilon^i{}_{jk}\omega^j\omega^k \equiv \frac{1}{2} R^i{}_{\mu\nu} dx^\mu dx^\nu\,, \\
T^i &= db^i + \varepsilon^i{}_{jk}\omega^j b^k \equiv \frac{1}{2}\,T^i{}_{\mu\nu} dx^\mu dx^\nu\,.
\end{aligned} \tag{2.1b}$$

General gravitational dynamics is defined by demanding the Lagrangian to be at most quadratic in field strengths [13]. Mielke and Baekler proposed a *topological* 3D model, with an action which is at most *linear* in field strengths [11,12]:

$$I = aI_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 + I_M\,, \tag{2.2a}$$

where I_M is a matter action, and

$$
\begin{aligned}
I_1 &= 2\int b^i R_i = -\int d^3x\, bR\,, \\
I_2 &= -\frac{1}{3}\int \varepsilon_{ijk} b^i b^j b^k = -2\int d^3x\, b\,, \\
I_3 &= \int \left(\omega^i d\omega_i + \frac{1}{3}\varepsilon_{ijk}\omega^i\omega^j\omega^k\right)\,, \\
I_4 &= \int b^i T_i\,.
\end{aligned}
\tag{2.2b}
$$

The first two terms are inspired by GR_Λ (where $a = 1/16\pi G$), I_3 is a Chern-Simons action for the Lorentz connection, and I_4 is an action of the translational Chern-Simons type. The MB model can be thought of as a generalization of GR_Λ ($\alpha_3 = \alpha_4 = 0$) to a topological gravity theory in Riemann-Cartan spacetime.

The field equations are obtained by variation with respect to the triad and connection. In the absence of matter, they take the form

$$
\begin{aligned}
&2aR_i + 2\alpha_4 T_i - \Lambda\varepsilon_{ijk} b^j b^k = 0\,, \\
&2\alpha_3 R_i + 2aT_i + \alpha_4\varepsilon_{ijk} b^j b^k = 0\,.
\end{aligned}
$$

Assuming $\alpha_3\alpha_4 - a^2 \neq 0$, these equations can be written in the simple form

$$
2T_i = A\varepsilon_{ijk} b^j b^k\,, \qquad 2R_i = B\varepsilon_{ijk} b^j b^k\,, \tag{2.3}
$$

where

$$
A = \frac{\alpha_3\Lambda + \alpha_4 a}{\alpha_3\alpha_4 - a^2}\,, \qquad B = -\frac{(\alpha_4)^2 + a\Lambda}{\alpha_3\alpha_4 - a^2}\,.
$$

Thus, the vacuum configuration of fields is characterized by constant torsion and constant curvature.

In Riemann-Cartan geometry one can use a well known identity to express the curvature 2-form R^{ij} in terms of its Riemannian piece $\tilde{R}^{ij}$ and the contortion [14], whereupon the second field equation in (2.3) can be transformed into an equivalent form,

$$
\tilde{R}^{ij} = -\Lambda_{\mathrm{eff}}\, b^i b^j\,, \qquad \Lambda_{\mathrm{eff}} \equiv B - \frac{1}{4}A^2\,, \tag{2.4}
$$

where Λ_{eff} is the effective cosmological constant. Regarded as an equation for metric, it implies that our spacetime is maximally symmetric [16]: for $\Lambda_{\mathrm{eff}} < 0$ ($\Lambda_{\mathrm{eff}} > 0$) it has the anti-de Sitter (de Sitter) form. This equation can be considered as an equivalent of the second equation in (2.3).

There are two interesting special cases of the general MB model: a) for $\alpha_3 = \alpha_4 = 0$, the Riemann-Cartan theory leads to Riemannian geometry ($A = 0$), in the context of which the Chern-Simons structure of gravity has been first discovered [3]; b) for $\alpha_3 = (\alpha_4)^2 + a\Lambda = 0$, the vacuum geometry becomes teleparallel ($B = 0$), but the related Chern-Simons structure remains the same as in Riemannian case [14,15].

B. Riemann-Cartan black hole

For a given Λ_{eff}, there is a simple method for finding classical solutions of the MB theory, defined by the following three-step procedure: (a) use equation (2.4) to find the metric of

our maximally symmetric space, (b) choose the triad field so as to satisfy the condition $ds^2 = b^i b^j \eta_{ij}$, and (c) use the first equation in (2.3) to define the related connection ω^i.

When the effective cosmological constant is negative,

$$\Lambda_{\text{eff}} \equiv -\frac{1}{\ell^2} < 0\,, \tag{2.5}$$

the condition for maximal symmetry (2.4) has a well known solution for the metric — the BTZ black hole [17]. Using the static coordinates $x^\mu = (t, r, \varphi)$, and units $4G = 1$, it is given as

$$\begin{aligned}
&ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\varphi + N_\varphi dt)^2\,,\\
&N^2 = \left(-2m + \frac{r^2}{\ell^2} + \frac{J^2}{r^2}\right)\,, \qquad N_\varphi = \frac{J}{r^2}\,.
\end{aligned}$$

The related triad field can be chosen in the simple form:

$$\begin{aligned}
&b^0 = N dt\,, \qquad b^1 = N^{-1} dr\,,\\
&b^2 = r\left(d\varphi + N_\varphi dt\right)\,.
\end{aligned} \tag{2.6a}$$

Then, the connection is obtained by solving the first equation in (2.3):

$$\begin{aligned}
&\omega^0 = N\left(\frac{A}{2}dt - d\varphi\right)\,, \qquad \omega^1 = N^{-1}\left(\frac{A}{2} + \frac{J}{r^2}\right) dr\,,\\
&\omega^2 = -\left(\frac{r}{\ell} - \frac{A\ell}{2}\frac{J}{r}\right)\frac{dt}{\ell} + \left(\frac{A}{2}r - \frac{J}{r}\right) d\varphi\,.
\end{aligned} \tag{2.6b}$$

Equations (2.6) define the *Riemann-Cartan* black hole [13]. For $A = 2/\ell$, and consequently $B = 0$, this solution reduces to the *teleparalell* black hole [14].

General solution of the MB model possessing maximal number of *global* symmetries defines the Riemann-Cartan AdS solution, AdS_3. It can be obtained from the black hole (2.6) by imposing $J = 0$, $2m = -1$.

In Riemannian geometry, AdS_3 is maximally symmetric space, hence it is locally isometric to any other solution having the same curvature and signature [16]. Since the same property remains valid in the MB model, our theory carries no local degrees of freedom.

C. Asymptotic AdS configuration

Dynamical properties of a theory are very sensitive to the choice of boundary conditions. They can be thought of as a mechanism for selecting a class of field configurations which have a special dynamical importance. Inspired by the results obtained in GR_Λ and the teleparallel 3D gravity, we focus our attention on the *asymptotic AdS configurations* of fields, which are determined by the following requirements [18,2]: (a) they are invariant under the action of the AdS group, (b) include the black hole configuration (2.6), and (c) the related asymptotic symmetries have well defined canonical generators.

At the moment, we focus our attention on the requirements (a) and (b). The asymptotics of the triad field $b^i{}_\mu$ that satisfies (a) and (b) is given by the relation

$$b^i{}_\mu = \begin{pmatrix} \frac{r}{\ell} + \mathcal{O}_1 & O_4 & O_1 \\ \mathcal{O}_2 & \frac{\ell}{r} + \mathcal{O}_3 & O_2 \\ \mathcal{O}_1 & \mathcal{O}_4 & r + \mathcal{O}_1 \end{pmatrix} . \tag{2.7a}$$

The arguments leading to this result are the same as those in Ref. [14]. The asymptotic form of the connection $\omega^i{}_\mu$ is defined with the help of the first equation in (2.3):

$$\omega^i{}_\mu = \begin{pmatrix} \frac{Ar}{2\ell} + \mathcal{O}_1 & \mathcal{O}_4 & -\frac{r}{\ell} + \mathcal{O}_1 \\ \mathcal{O}_2 & \frac{A\ell}{2r} + \mathcal{O}_3 & \mathcal{O}_2 \\ -\frac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_4 & \frac{Ar}{2} + \mathcal{O}_1 \end{pmatrix} . \tag{2.7b}$$

The improved conditions $\omega^0{}_1, \omega^2{}_1 = \mathcal{O}_4$ are in agreement with the constraints of the theory [14,15].

A direct verification of the third condition (c) would demand a rather lengthy canonical analysis. Instead of that, we shall first derive the Chern-Simons formulation of our theory, whereupon the condition (c) will follow straightforwardly.

III. CHERN-SIMONS FORMULATION

Our experience with GR_Λ and the teleparallel 3D theory shows that the dynamical structure of 3D gravity becomes much simpler if it can be formulated as an ordinary gauge theory of the Chern-Simons type [3,15]. In this section we show that the general topological Riemann-Cartan 3D gravity (2.2) can also be represented as a Chern-Simons theory.

A. Discovering $SL(2,R) \times SL(2,R)$ gauge symmetry

Every Poincaré gauge theory is by construction invariant under the local Poincaré transformations (2.1a). We shall see that, under certain conditions, this symmetry can be represented as an $SL(2,R) \times SL(2,R)$ gauge symmetry.

Let us begin by introducing new variables and gauge parameters by the relations

$$A^i{}_\mu = \omega^i{}_\mu + q b^i{}_\mu \,, \qquad \bar{A}^i{}_\mu = \omega^i{}_\mu + \bar{q} b^i{}_\mu \,, \tag{3.1a}$$
$$u^i = -\theta^i - \xi^\mu A^i{}_\mu \,, \qquad \bar{u}^i = -\theta^i - \xi^\mu \bar{A}^i{}_\mu \,, \tag{3.1b}$$

with $q \neq \bar{q}$. Expressed in terms of these, the local Poincaré symmetry (2.1a) takes the form:

$$\delta_0 A^i{}_\mu = \nabla_\mu u^i + \xi^\rho F^i{}_{\mu\rho} \,, \qquad \delta_0 \bar{A}^i{}_\mu = \bar{\nabla}_\mu \bar{u}^i + \xi^\rho \bar{F}^i{}_{\mu\rho} \,. \tag{3.2}$$

where we use the notation

$$\nabla_\mu u^i = \partial_\mu u^i + \varepsilon^i{}_{jk} A^j{}_\mu u^k \,,$$
$$F^i{}_{\mu\nu} = \partial_\mu A^i{}_\nu - \partial_\nu A^i{}_\mu + \varepsilon^i{}_{jk} A^j{}_\mu A^k{}_\nu \,,$$

and similarly for $\bar{\nabla}$ and $\bar{F}$. Thus, local Poincaré symmetry takes the form of an internal gauge symmetry, provided the field equations of our theory are equivalent to

$$F^i{}_{\mu\nu} = 0\,, \qquad \bar{F}^i{}_{\mu\nu} = 0\,. \tag{3.3}$$

The internal symmetry can be identified with $SL(2,R) \times SL(2,R)$, as follows from the form of the structure constants $\varepsilon^i{}_{jk}$ and the metric η_{ij} (see, e.g. Ref. [15] for details). Can we really transform equations (2.3) to the form (3.3)? In order to verify this, we start from the identities

$$\begin{aligned} F^i{}_{\mu\nu} &= R^i{}_{\mu\nu} + qT^i{}_{\mu\nu} + q^2\varepsilon^i{}_{jk}b^j{}_\mu b^k{}_\nu\,, \\ \bar{F}^i{}_{\mu\nu} &= R^i{}_{\mu\nu} + \bar{q}T^i{}_{\mu\nu} + \bar{q}^2\varepsilon^i{}_{jk}b^j{}_\mu b^k{}_\nu\,, \end{aligned}$$

which imply that equations (3.3) can be rewritten as

$$T_{ijk} = -(q+\bar{q})\varepsilon_{ijk}\,, \qquad R_{ijk} = q\bar{q}\varepsilon_{ijk}\,. \tag{3.4}$$

Consequently, equations (3.3) coincide with (2.3) if $A = -(q+\bar{q})$, $B = q\bar{q}$, or equivalently:

$$\begin{aligned} 2q &= -A + \sqrt{A^2 - 4B}\,, \\ 2\bar{q} &= -A - \sqrt{A^2 - 4B}\,. \end{aligned} \tag{3.5}$$

Demanding that the parameters q and $\bar{q}$ be real and different from each other, we obtain the following restrictions on A and B:

$$A^2 - 4B > 0 \quad \Leftrightarrow \quad \Lambda_{\text{eff}} \equiv B - \frac{1}{4}A^2 < 0\,. \tag{3.6}$$

– In the AdS sector of the MB theory, i.e. for $\Lambda_{\text{eff}} < 0$, the gravitational field equations are equivalent to the Chern-Simons equations (3.3), and local Poincaré symmetry coincides (on shell) with the $SL(2,R) \times SL(2,R)$ gauge symmetry.

B. Chern-Simons form of the action

Having found two independent $SL(2,R)$ gauge symmetries at the level of field equations, we now wish to find out, following Witten [3], whether the action of 3D gravity can be represented as a combination of two pieces, each of which depends only on one of two independent gauge fields, A or $\bar{A}$.

Starting from the Chern-Simons Lagrangian,

$$\mathcal{L}_{\text{CS}}(A) = A^i dA_i + \frac{1}{3}\varepsilon_{ijk}A^iA^jA^k\,, \tag{3.7}$$

where $A^i = A^i{}_\mu dx^\mu$, $A_i = \eta_{ij}A^j$, we can use the expressions (3.1a) for A^i i $\bar{A}^i$ and obtain the relation

$$\mathcal{L}_{\text{CS}}(A) = \mathcal{L}_{\text{CS}}(\omega) + 2qb^iR_i + q^2b^iT_i + \frac{1}{3}q^3\varepsilon_{ijk}b^ib^jb^k + qd(b^i\omega_i)\,.$$

This result leads directly to the important identity

$$\kappa_1\mathcal{L}_{\rm CS}(A)-\kappa_2\mathcal{L}_{\rm CS}(\bar{A}) = 2ab^iR_i-\frac{1}{3}\Lambda\varepsilon_{ijk}b^ib^jb^k \\ +\alpha_3\mathcal{L}_{\rm CS}(\omega)+\alpha_4 b^iT_i+ad(b^i\omega_i)\,, \tag{3.8a}$$

where

$$\begin{aligned} &a=\kappa_1q-\kappa_2\bar{q}\,, \qquad \Lambda=-(\kappa_1q^3-\kappa_2\bar{q}^3)\,, \\ &\alpha_3=\kappa_1-\kappa_2\,, \qquad \alpha_4=\kappa_1q^2-\kappa_2\bar{q}^2\,. \end{aligned} \tag{3.8b}$$

Comparing with (2.2) we obtain the final result:

$$\kappa_1\mathcal{L}_{\rm CS}(A)-\kappa_2\mathcal{L}_{\rm CS}(\bar{A}) = \mathcal{L}_G+ad(b^i\omega_i)\,, \tag{3.9}$$

where $\mathcal{L}_G$ denotes the gravitational Lagrangian in (2.2).

Let us note that under the boundary conditions (2.7) the Chern-Simons actions $I_{\rm CS}[A]$ and $I_{\rm CS}[\bar{A}]$ do not have well defined functional derivatives [19,20]. However, without loss of generality, these conditions can be refined to ensure the needed differentiability, as we shall see in the next subsection. After that, equation (3.9) can be written in the simple form

$$\kappa_1I_{\rm CS}[A]-\kappa_2I_{\rm CS}[\bar{A}]=\tilde{I}_G\,, \tag{3.10a}$$

where $\tilde{I}_G$ is the improved (differentiable) gravitational MB action:

$$\tilde{I}_G=I_G+a\int d(b^i\omega_i)\,. \tag{3.10b}$$

– The improved gravitational action of the Riemann-Cartan 3D gravity is equal to a linear combination of two Chern-Simons actions.

The role of coefficients κ_1 and κ_2 is to define central charges of the theory [5].

The parameters $(a,\Lambda,\alpha_3,\alpha_4)$ are functionally independent of $(\kappa_1,\kappa_2,q,\bar{q})$ if the Jacobian of the transformations does not vanish. From

$$\det\frac{\partial(a,\Lambda,\alpha_3,\alpha_4)}{\partial(\kappa_1,\kappa_2,q,\bar{q})}=-\kappa_1\kappa_2(q-\bar{q})^4\,,$$

we see that the transformation of parameters is regular for $q\neq\bar{q}$. Using equations (3.8b), we can explicitly express $(\kappa_1,\kappa_2,q,\bar{q})$ in terms of $(a,\Lambda,\alpha_3,\alpha_4)$:

$$\begin{aligned} &q=-\frac{A}{2}+\frac{1}{\ell}\,, \qquad \bar{q}=-\frac{A}{2}-\frac{1}{\ell}\,, \\ &\kappa_1-\kappa_2=\alpha_3\,, \qquad \kappa_1+\kappa_2=\ell\left(a+\frac{A}{2}\,\alpha_3\right)\,. \end{aligned} \tag{3.11}$$

The above relation clarifies the role of four parameters appearing in the action (2.2). In particular, our gravitational theory with $\alpha_3\neq0$ (and boundary conditions (2.7)) has conformal symmetry with *two different central charges*:

$$c_i=12\cdot4\pi\kappa_i \qquad (i=1,2)\,. \tag{3.12}$$

Table 1 illustrates two particular cases belonging to the complementary sector $\alpha_3=0$.

Table 1. Particular cases of the MB model with $\Lambda_{\text{eff}} < 0$

Conditions	Geometry	q	$\bar{q}$	κ_1	κ_2
$\alpha_3 = 0\,,\ A = 0$	Riemannian	$1/\ell$	$-1/\ell$	$a\ell/2$	$a\ell/2$
$\alpha_3 = 0\,,\ B = 0$	Teleparallel	0	$-2/\ell$	$a\ell/2$	$a\ell/2$

The first case corresponds to GR_Λ [3], the second one to the teleparallel theory [14]. In both cases we have two copies of the same central charge,

$$c_1 = c_2 = 48\pi\frac{a\ell}{2} = \frac{3\ell}{2G}\,.$$

We can now return to the condition (c) formulated in subsection II.C. Using the relations (3.10) and the known results concerning the canonical structure of the Chern-Simons theory, see e.g. Refs. [5,10], one can conclude that the asymptotic symmetries of the theory defined by the action $\tilde{I}_G$ and the boundary conditions (2.7) have well defined canonical generators.

C. Asymptotic conditions

Since the dynamical content of Chern-Simons theory is influenced by the form of boundary conditions, we shall now investigate the corresponding behavior of A and $\bar{A}$.

The asymptotic conditions (2.7) for the gravitational variables ω^i and b^i, in conjunction with the values of q and $\bar{q}$ given in (3.11), lead to the following conditions on A^i and $\bar{A}^i$:

$$A^i{}_\mu = \begin{pmatrix} \dfrac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_4 & -\dfrac{r}{\ell} + \mathcal{O}_1 \\ \mathcal{O}_2 & \dfrac{1}{r} + \mathcal{O}_3 & \mathcal{O}_2 \\ -\dfrac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_4 & \dfrac{r}{\ell} + \mathcal{O}_1 \end{pmatrix}, \tag{3.13a}$$

$$\bar{A}^i{}_\mu = \begin{pmatrix} -\dfrac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_4 & -\dfrac{r}{\ell} + \mathcal{O}_1 \\ \mathcal{O}_2 & -\dfrac{1}{r} + \mathcal{O}_3 & \mathcal{O}_2 \\ -\dfrac{r}{\ell^2} + \mathcal{O}_1 & \mathcal{O}_4 & -\dfrac{r}{\ell} + \mathcal{O}_1 \end{pmatrix}. \tag{3.13b}$$

Using the light-cone notation, $x^\pm = t/\ell \pm \varphi$, $A^\pm = A^0 \pm A^2$, the conditions for A^i can be rewritten in the form:

$$\begin{aligned} A^1 &= \frac{dr}{r} + \mathcal{O}_2\,, \qquad A^+ = \mathcal{O}_1\,, \\ A^- &= \frac{2r}{\ell}dx^- + \mathcal{O}_1\,, \end{aligned} \tag{3.14a}$$

with the additional restrictions on A_1:

$$A_1^\pm = \mathcal{O}_4\,, \qquad A_1^1 = \frac{1}{r} + \mathcal{O}_3\,. \tag{3.14b}$$

Similarly, we find for $\bar{A}^i$:

$$\bar{A}^1 = -\frac{dr}{r} + \mathcal{O}_2 \,, \qquad \bar{A}^- = \mathcal{O}_1 \,,$$
$$\bar{A}^+ = -\frac{2r}{\ell} dx^+ + \mathcal{O}_1 \,, \tag{3.15a}$$

together with

$$\bar{A}_1^{\pm} = \mathcal{O}_4 \,, \qquad \bar{A}_1^1 = -\frac{1}{r} + \mathcal{O}_3 \,. \tag{3.15b}$$

Having found the boundary conditions for A and $\bar{A}$, we now return to the interpretation of the action $\tilde{I}_G$ in (3.10). The variation of the Chern-Simons action takes the form

$$\delta I_{\mathrm{CS}}[A] = \int_{\mathcal{M}} \delta A^i F_i + \int_{\partial\mathcal{M}} A^i \delta A_i \,,$$

hence it has well defined functional derivatives if the boundary term vanishes. Relations (3.14) and (3.15) yield the conditions on A_+ and $\bar{A}_-$,

$$A_+^1 = \mathcal{O}_2 \,, \quad A_+^- = \mathcal{O}_1 \,, \quad A_+^+ = \mathcal{O}_1 \,,$$
$$\bar{A}_-^1 = \mathcal{O}_2 \,, \quad \bar{A}_-^+ = \mathcal{O}_1 \,, \quad \bar{A}_-^- = \mathcal{O}_1 \,,$$

which are seen to be insufficient for the differentiability of $I_{\mathrm{CS}}[A]$ and $I_{\mathrm{CS}}[\bar{A}]$. Fortunately, we can get rid of the problem in a simple manner — by adopting the refined conditions

$$A_+^+ = \mathcal{O}_3 \,, \qquad \bar{A}_-^- = \mathcal{O}_3 \,,$$

compatible with the field equations. Equation (3.10a) then implies that $\tilde{I}_G$ is also differentiable.

The boundary conditions (3.14) and (3.15) have the same form as in the teleparallel case [15], the only difference being the different value of the constant ℓ. Following the same line of arguments, based on the form of field equations and boundary conditions, one verifies that the complete dynamics is located at the boundary, and is described by two chiral fields (functions of only x^+ or x^-). According to the analysis of the preceding subsection,

– general dynamical evolution of these boundary fields follows the rules of a conformal field theory possessing two different classical central charges.

IV. CONCLUDING REMARKS

In this paper we show that the general Riemann-Cartan topological 3D gravity (2.2) can be formulated as an $SL(2,R) \times SL(2,R)$ Chern-Simons gauge theory, provided we stay in its AdS sector, defined by the condition $\Lambda_{\mathrm{eff}} < 0$. More precisely, we show that

i) the improved gravitational action (2.2) is represented as a linear combination of two independent Chern-Simons actions, equation (3.10),

and also that the gravitational AdS boundary conditions (2.7) transform into the standard Chern-Simons boundary conditions (3.13). As a consequence,

ii) boundary dynamics of the general Riemann-Cartan gravity (2.2) is described by a conformal field theory with two different classical central charges,

in contrast to the cases of GR_Λ and the teleparallel theory [3,14].

One should note that in GR_Λ and the teleparallel gravity, where $c_1 = c_2$, the non-trivial boundary components of A and $\bar{A}$ are associated with a single Liouville field [4,14]. The corresponding effective two-dimensional action (the Liouville action) correctly reproduces both the field equations and the central charge of the associated gravity theory. In the general MB theory, the boundary fields can still be associated with a single Liouville field subject to the Liouville field equations. The Liouville action, however, does not correctly reproduce the needed two central charges. It would be interesting to find the form of an effective 2-dimensional action that would correctly reproduce both the field equations and the central charges of the general MB theory.

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